KEPLER'S LAWS

Copernicus was the person who introduced the idea that the central body of our planetary system was sun rather than Earth. Galileo's construction of a telescope enabled him to discover direct visual evidence supporting 'Copernicus theory'. Kepler later on confirmed it by putting it on a solid mathematical background. Kepler announced two of his laws in 1609 and the third one 10 years later.

Kepler's laws of planetary motion are as follows:-

1) **The law of elliptical orbit**:- Every planets revolves around the sun in elliptical orbits with sun situated at one of its foci.

Since the focus of an ellipse is not equidistant from the points on the orbit, the distance of the planet from the sun varies b/w a certain minimum and a maximum value. Rotation is the reason for a change of season and repetation of the season after one year.

Proof:- As planet revolves around the sun under gravitational force $f(r) = -\frac{K}{r^2}$, where $K = G m_1 m_2$

$$V(r) = -\int f(r)\, dr$$

Then potential energy, $V(r) = \frac{-K}{r}$

Using this value in the expression,

$$d\theta = \frac{L\, dr}{2\mu r^2 \sqrt{\frac{2}{\mu}\left(E - V - \frac{L^2}{2\mu r^2}\right)}}$$

we get, $$d\theta = \frac{L\, dr}{2\mu r^2 \sqrt{\frac{2}{\mu}\left[E + \frac{K}{r} - \frac{L^2}{2\mu r^2}\right]}}$$

Integrating we get,

$$\therefore \theta = \int \frac{L\, dr}{4r^2 \sqrt{2/\mu\,(E + K/r - L^2/2\mu r^2)}} + \theta_0 \quad \text{——— (1)}$$

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where θ_0 = const. of integration

If we put $u = \frac{1}{r}$ i.e. $r = \frac{1}{u} \Rightarrow dr = -\frac{1}{u^2} du$. Then eq^n (1) be

$$\theta = -\int \frac{L\, dr}{\sqrt{\left(\frac{2\mu E}{L^2} + \frac{2\mu K u}{L^2} - u^2\right)^{1/2}}} + \theta_0$$

$$\Rightarrow \theta = \theta_0 - \int \frac{du}{\left(\frac{2\mu E}{L^2} + \frac{2\mu K u}{L^2} - u^2\right)^{1/2}} \quad \text{— (2)}$$

Using standard value of integral,

$$\int \frac{dx}{\sqrt{a + bx + cx^2}} = \frac{1}{\sqrt{-c}} \cos^{-1}\left(-\frac{b + 2cx}{\sqrt{b^2 - 4ac}}\right)$$

Comparing with $\int \frac{dx}{\sqrt{\frac{2\mu E}{L^2} + \frac{2\mu K u}{L^2} - u^2}}$

we get $c = -1$, $a = 2\mu E/L^2$, $b = 2\mu K/L^2$

Then $\int \frac{du}{\sqrt{\frac{2\mu E}{L^2} + \frac{2\mu K u}{L^2} - u^2}} = \cos^{-1}\left[\frac{-2\mu K/L^2 + 2u}{\sqrt{\frac{4\mu^2 K^2}{L^4} + 4\left(\frac{2\mu E}{L^2}\right)}}\right]$

$$= \cos^{-1}\left[\frac{2\mu K/L^2\left(\frac{uL^2}{\mu K} - 1\right)}{\left\{\frac{4\mu^2 K^2}{L^4}\left(1 + \frac{2EL^2}{\mu K^2}\right)\right\}^{1/2}}\right]$$

$$= \cos^{-1}\frac{\left(\frac{uL^2}{\mu K} - 1\right)}{\sqrt{1 + \frac{2EL^2}{\mu K^2}}} \quad \text{— (3)}$$

Using eq^n (3) in eq^n (2) we get:

$$\theta = \theta_0 - \cos^{-1}\frac{\left(\frac{u\mu}{\mu K} - 1\right)}{\sqrt{1 + \frac{2EL^2}{\mu K^2}}}$$

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$$\Rightarrow \cos(\theta-\theta_0) = \frac{\left(\frac{\mu L^2}{\mu k} - 1\right)}{\sqrt{1+\frac{2EL^2}{\mu k^2}}} \qquad \left[\because \cos(\theta-\theta_0) = \cos(\theta_0-\theta)\right]$$

$$\Rightarrow \frac{\mu L^2}{\mu k} - 1 = \sqrt{\left(1+\frac{2EL^2}{\mu k^2}\right)}\cos(\theta-\theta_0)$$

$$\Rightarrow u = \frac{\mu k}{L^2}\left[1+\sqrt{\left(1+\frac{2EL^2}{\mu k^2}\right)}\cos(\theta-\theta_0)\right] \quad \text{—— (4)}$$

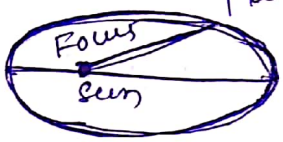
Let's substitute $\epsilon = \sqrt{1+2EL^2/\mu k^2}$ —— (5)

and $c = \mu k/L^2$ —— (6)

Eq^n (4) becomes $u = c\left[1+\epsilon\cos(\theta-\theta_0)\right]$

i.e. $\frac{1}{r} = c\left[1+\epsilon\cos(\theta-\theta_0)\right]$ —— (7)

which is the ~~required~~ equation of the conic section with ϵ as eccentricity and focus as the origin.



From eq^n (5) we have,

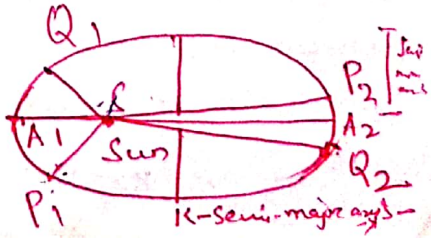
$$\frac{2EL^2}{\mu k^2} = \epsilon^2 - 1$$

$$\Rightarrow E = -\frac{\mu k^2}{2L^2}(1-\epsilon^2)$$ which is -ve if $\epsilon < 1$.

This corresponds to elliptical orbit. So the planet moves around the sun in elliptical orbit with sun at one focus. This is Kepler's 1st law.

(ii) Law of areal Velocities :- A planet moves round the sun in such a way that it's areal velocity is constant i.e. (the line joining the planet with the sun sweeps equal areas in equal intervals of time).

Proof. Let t be the time taken by the planet to go from P_1 to Q_1 so that the line SP_1 traverses an area P_1SQ_1. While going from P_2 to Q_2, the planet moves in such a way that



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Area P_2SQ_2 = Area P_1SQ_1

Since $SP_2 > SP_1$ $\therefore$ $P_2Q_2 < P_1Q_1$

P_1Q_1 and P_2Q_2 are the distances travelled, along the orbit, in same time. From this, we can conclude that the orbital velocity of a planet is not uniform. It is largest when the planet is nearest to sun and is least when the planet is at maximum distances away from sun.

Proof:- Let the planet move from P to Q in a time Δt so that its position vector changes from $\vec{r}$ to $\vec{r} + \Delta\vec{r}$.

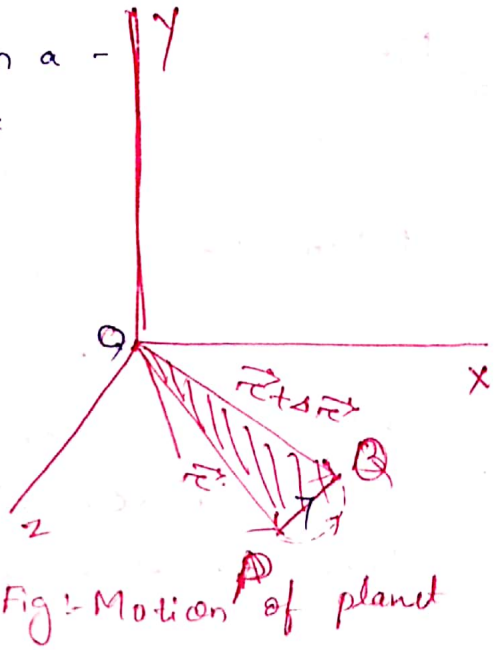

Fig:- Motion of planet

$\therefore$ $\vec{PQ} = \vec{r} + \Delta\vec{r} - \vec{r} = \Delta\vec{r}$

Let $\Delta\vec{A}$ be the area swept by its radius vector.

From fig: $\Delta\vec{A} = \frac{1}{2}(\vec{OP} \times \vec{PQ})$

$\Rightarrow \Delta\vec{A} = \frac{1}{2}\vec{r} \times \Delta\vec{r}$

But $\vec{v} = \frac{\Delta\vec{r}}{\Delta t} \Rightarrow \Delta\vec{r} = \vec{v}\Delta t$

$\therefore$ $\Delta\vec{A} = \frac{1}{2}\vec{r} \times \vec{v}\,\Delta t$ —— (8)

If $\vec{p}$ is the momentum of the planet,

$\vec{v} = \vec{p}/m$

$\therefore$ $\Delta\vec{A} = \frac{1}{2}\vec{r} \times \left(\frac{\vec{p}}{m}\Delta t\right)$

$\Rightarrow \Delta\vec{A} = \frac{1}{2m}(\vec{r} \times \vec{p})\,\Delta t$

$\Rightarrow \frac{\Delta\vec{A}}{\Delta t} = \frac{1}{2m}(\vec{L})$ —— (9)

But as a gravitational force is a central force and from force law is the same as given we know that angular momentum is conserved.

$\therefore$ Areal velocity $= \frac{\text{const.}}{2m} = \text{const.}$ —— (10)

This is Kepler's 2nd law.

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iii) Law of time periods:- 'The harmonic law':-

A planet moves round the sun in such a way that the square of it's time period is proportional to the cube of semi-major axis of its elliptical orbit.

$$\text{or } T^2 \propto R^3$$

If T_1 and T_2 are the time periods of two planets having R_1 and R_2 respectively, as the semi-major axes of the orbits of two planets, then

$$\frac{T_1^2}{T_2^2} = \frac{R_1^3}{R_2^3}$$

Proof:- Let the planets moves around the sun in circular orbits.

Consider earth of mass M, revolving around the sun of mass m. C is the centre of mass of the system.

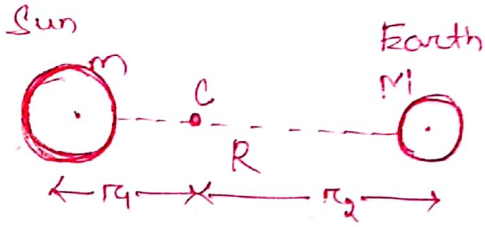

Fig: Earth revolving around the sun

Take moments of C,

$$mr_1 = Mr_2$$

$$\text{or } r_1/r_2 = M/m$$

Add 1 to both sides,

$$r_1/r_2 + 1 = M/m + 1$$

$$\Rightarrow \frac{r_1 + r_2}{r_2} = \frac{M+m}{m}$$

$$\Rightarrow r_2 = \frac{mR}{M+m} \quad \text{---(1)} \qquad [\because R = r_1 + r_2]$$

As earth revolves around the sun, the gravitational force F_g b/w them provides the necessary centripetal force F_c.

$$F_g = \frac{GMm}{R^2}$$

$$F_c = \frac{Mv^2}{r_2}$$

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Using $F_g = F_c$

$$\Rightarrow \frac{GMm}{R^2} = \frac{Mv^2}{r_2}$$

$$\Rightarrow \frac{GM}{R^2} = \frac{v^2}{r_2} \quad \text{——(12)}$$

If T = time period of rotation of earth,

$$v = \frac{2\pi r_2}{T}$$

Substituting for v, $\frac{GM}{R^2} = \left(\frac{2\pi r_2}{T}\right)^2 \times \frac{1}{r_2} = \frac{4\pi^2}{T^2} r_2$

Substituting for r_2 from eqn (i),

$$\frac{Gm}{R^2} = \frac{4\pi^2}{T^2} \times \frac{mR}{m+M}$$

$$\Rightarrow T^2 = \frac{4\pi^2}{G(M+m)} R^3 \quad \text{——(13)}$$

Since $\frac{4\pi^2}{G(M+m)}$ is a const. quantity.

$$T^2 \propto R^3$$

This establishes Kepler's law of time periods.

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